The 1+1-dimensional Kardar-Parisi-Zhang equation and its universality class

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Abstract. We explain the exact solution of the 1+1 dimensional Kardar-Parisi-Zhang equation with sharp wedge initial conditions. Thereby it is confirmed that the continuum model belongs to the KPZ universality class, not only as regards to scaling exponents but also as regards to the full probability distribution of the height in the long time limit.

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1 Introduction

Over the past year there has been significant advances in the understanding of surface growth in 1+1 dimensions, both on the experimental and theoretical side. The experiments on droplet growth for a thin film of turbulent liquid crystal were presented at StatPhys24 by K.A. Takeuchi [1]. We will report here on the theoretical findings, but use the occasion to put the results in a wider context. We also take the liberty to comment on the experimental situation.

In their seminal 1986 paper [2] Kardar, Parisi, and Zhang (KPZ) proposed an evolution equation for growing interfaces which in 1+1 dimensions, of interest here, reads

$$\frac{\partial}{\partial t}h = \tfrac{1}{2}\lambda\big(\frac{\partial}{\partial x}h\big)^2 + \nu\frac{\partial^2}{\partial x^2}h + \sqrt{D}\eta\,. \tag{1}$$

(1) is a stochastic evolution equation for the height function $h(x,t)$, hence has to be supplemented by the appropriate initial conditions. The height h depends on the spatial location $x \in \mathbb{R}$ and on time $t \in \mathbb{R}_+$. $\tfrac{1}{2}\lambda(\partial h/\partial x)^2$ is the nonlinear growth velocity depending quadratically on the local slope. The Laplacian smoothens the height profile with relaxation coefficient $\nu > 0$. $\eta(x,t)$ is space-time white noise, *i.e.* η is Gaussian with mean zero and covariance $\langle \eta(x,t)\eta(x',t')\rangle = \delta(x - x')\delta(t - t')$. $\sqrt{D}$ is the intensity of the noise and $\sqrt{D}\eta$ models the random nucleation events at the interface. We consider only the infinite line without boundary conditions. Finite geometry and possibly boundary sources are of physical interest. The corresponding large deviations have been studied by Brunet and Derrida [3]. In this article our main focus is on typical fluctuations. Their finite volume corrections have been little explored so far.

Already in their original contribution, KPZ noted that (1) can be transformed to a linear equation at the expense of turning the additive noise η into multiplicative noise. More precisely, one introduces the Cole-Hopf transform

$$Z(x,t) = \exp\big[(\lambda/2\nu)h(x,t)\big]\,. \tag{2}$$

Then Z satisfies

$$\frac{\partial}{\partial t}Z(x,t) = \nu\frac{\partial^2}{\partial x^2}Z(x,t) + (\lambda\sqrt{D}/2\nu)\eta(x,t)Z(x,t)\,. \tag{3}$$

To "solve" (3) we introduce the auxiliary standard Brownian motion $b(t)$, $t \geq 0$, with variance t. Then, by the Feynman-Kac formula,

$$Z(x,t) = \mathbb{E}_x\Big(\exp\big[\alpha\int_0^{2\nu t}ds\eta(b(s),s)\big]\exp\big[(\lambda/2\nu)h(b(2\nu t),0)\big]\Big) \tag{4}$$

with $\alpha = (2\nu)^{-3/2}\lambda D^{1/2}$. To distinguish from the white noise average denoted by $\langle\cdot\rangle$, the average over the Brownian motion starting at x is denoted here by $\mathbb{E}_x(\cdot)$. $h(x,0)$ is the initial height profile. According to (3) the polymer has inverse stiffness ν, which for notational convenience has been absorbed in the time scale of $b(t)$.

$Z(x,t)$ can be viewed as the random partition function of the directed polymer $b(s)$, $0 \le s \le 2\nu t$. The polymer is subject to the random potential η with strength α. To compute the potential energy of the polymer one simply has to sum η along the location of the polymer chain. Inverting (2), the height is the random free energy of the polymer. Thus the KPZ equation is equivalent to a model in the equilibrium statistical mechanics of disordered systems. Since the free energy valleys are in two-dimensional space-time, the directed polymer is easier to visualize than the high-dimensional configuration space of Ising spin glasses.

We remark that the KPZ equation and the Cole-Hopf transformation generalize in the obvious way from $1+1$ to $d+1$ dimensions.

There are books, comprehensive reviews, and a large number of original articles on growth models in the KPZ class [4, 5, 6, 7]. Of course, we cannot provide here a comprehensive summary. But very schematically, it seems reasonable to distinguish two in essence disjoint lines of research.

• Scaling theory, critical exponents, and phase diagram in general dimension, starting with KPZ in 1986,
• exact probability density functions (pdf) for growth models in $1+1$ dimensions, starting with Johansson [8] in 2000.

We comment on a few general features.

Scaling theory, phase diagram. Interface motion in the KPZ class is characterized by local growth rules with single growth events being statistically independent in space-time. There is no mass transport along the interface. The respective bordering bulk phases are statistically homogeneous and non-critical. As a rule, in the course of time the interface develops a statistically self-similar structure, when viewed on sufficiently coarse scales. The self-similar structure has been amply confirmed, mostly by Monte-Carlo simulations of various discretized models, for which there are many options: space can be discretized to a lattice $\mathbb{Z}$, the height may take only integer values, and one can implement updates in discrete time steps. Slightly more distant from (1) would be models as the off-lattice Eden growth model, where hard balls of equal size are added at random one by one to the current cluster of balls [9]. In fact, convincing numerical solutions of the $1+1$ dimensional KPZ equation are only fairly recent [10]. All these models constitute the KPZ universality class. One expects that in the infrared scaling, *i.e.* for large distances and long times, essentially all microscopic details become irrelevant and universal statistical laws emerge.

Amongst the many advances the most important findings can be summarized as follows.

(i) Phase diagram. For dimension $d \le 2$ there is only a strong coupling phase, excluding exceptional points where the effective coupling constant vanishes. For $d > 2$ there is a weak coupling and strong coupling phase. In the weak coupling phase the nonlinearity is irrelevant and fluctuations are according to the linear ($\lambda = 0$) Edwards-Wilkinson equation. Over the last ten years probabilists have proved in detail many of these claims. They mostly consider a directed polymer, where $b(t)$ is

replaced by a discrete time random walk and $\eta(x,t)$ by a collection of independent random potentials $V(j,t)$, $j \in \mathbb{Z}$, $t \in \mathbb{Z}_+$. In particular, it is proved that in the weak coupling regime the directed polymer has the statistics of a Brownian motion with a diffusion constant computed from the annealed version, see [11] for an overview and [12] for recent progress. In fact, the bounds on the phase diagram are fairly sharp. From the perspective of spin glasses a natural quantity to consider is the ratio $\langle Z(x,t)^2 \rangle / \langle Z(x,t) \rangle^2$. For $d > 2$, at small coupling the ratio stays bounded, while beyond a critical coupling, $\lambda_c^{(2)}$, it diverges exponentially. Based on theoretical arguments and numerical simulations it has been claimed that the true critical coupling $\lambda_c = \lambda_c^{(2)}$ [13]. Rigorous bounds establish $\lambda_c > \lambda_c^{(2)}$, although with a fairly small difference [14, 15].

(ii) Critical exponents. There is a static critical exponent, χ, which governs the height-height correlations in the steady state at fixed slope and a dynamical critical exponent, z. Both are related through

$$z + \chi = 2 \,. \tag{5}$$

For $d = 1$, in the steady state of the KPZ equation the slope $\partial h / \partial x$ is distributed according to spatial white noise, hence $\chi = \frac{1}{2}$ and $z = 3/2$. For $d > 1$, extensive numerical simulations are available. In the analysis of the numerical data one has to assume, implicitly or explicitly, a particular form of the finite size scaling, for which no systematic theory is available. This is one essential limiting factor in the attempt to compare with theories making definite predictions for the value of the exponents.

(iii) Nonuniversal constants. All microscopic details of either an experimental realization or of a particular theoretical model are subsumed in three nonuniversal constants [16, 17]. One considers a particular direction along which the interface has a non-random time-independent macroscopic slope u. Then the first coefficient is simply the steady state growth velocity v_∞ reached for long times. The second one is the effective coupling constant, defined by the Hessian $(\partial^2 / \partial u_i \partial u_j) v_\infty$. The KPZ theory assumes that the eigenvalues of the Hessian are either all strictly positive or all strictly negative. (If the eigenvalues have a different signature the fluctuations will be Gaussian [18, 19].) The third coefficient, A, is the amplitude of the steady state correlation function at slope u, i.e. $\langle (h(x) - h(x'))^2 \rangle \cong A|x - x'|^{2\chi}$ for large $|x - x'|$. In general, all nonuniversal coefficients depend on the slope u.

In the present context, the central unsolved problem is the upper critical dimension. One prediction, based on asymptotic expansions, is that z increases with d and reaches the asymptotic value 2 as $d \to \infty$. Other field theory based approaches yield the upper critical dimension $d = 4$. For $d > 4$ one finds $z = 2$, like in the weak coupling phase, but there are other statistical properties which still distinguish between weak and strong coupling phase. We refer to [20, 21, 22] and the recent letter by Canet *et al.* [23] with references to earlier literature. $\diamondsuit$

Exact solutions in 1+1 dimensions. Ulam proposed the following combinatorial

problem: One considers a random permutation $p(1) \ldots p(N)$ of $1 \ldots N$. Any given permutation will have increasing subsequences, *e.g.* 6 4 1 5 2 3 has the increasing subsequences 4 5, 1 2 3 and others. Ulam asked for the random length, ℓ_N, of the longest increasing subsequence (there could be several ones with the same length). One finds that typically $\ell_N = \mathcal{O}(\sqrt{N})$ for large N. So the more detailed question deals with the fluctuations relative to $\sqrt{N}$. In a famous contribution, Baik, Deift, and Johansson [24] established that the fluctuations are of order $N^{1/6}$. While the exponent has been anticipated from numerical simulations, the real surprise was that the pdf of the fluctuations turned out to be identical to the Tracy-Widom distribution known from the Gaussian unitary ensemble (GUE), thus establishing a completely unexpected link to random matrix theory. Johansson [8] extended these results to the single step growth model (alias TASEP, the totally asymmetric simple exclusion process). In this model space is discrete, $j \in \mathbb{Z}$, and a height variable $h(j,t)$ takes only integer values satisfying the single step constraint $|h(j+1,t)-h(j,t)| = 1$. In the stochastic update, independently at each local minimum the height is increased by 2 with probability dt and stays put with probability $1 - dt$. As initial condition we set $h(j,0) = |j|$. Johansson proved that for the height at origin it holds

$$h(0,t) = \tfrac{1}{2}t + 2^{-1/3}t^{1/3}\xi_{\mathrm{TW}} \tag{6}$$

for large t. Here ξ_{TW} is a GUE Tracy-Widom distributed random variable, as will be discussed further below. As predicted by the KPZ theory the fluctuations of the height are of order $t^{1/3}$. Also, the numerical coefficients in (6) are the nonuniversal constants explained in *(iii)* above.

In [25] it was shown that Ulam's problem is equivalent to the polynuclear growth model (PNG) with droplet geometry, *i.e.* the height function has typically the shape of a semicircle. As a surprising consequence, it was realized that while the size of the fluctuations is always of order $t^{1/3}$, the statistics still carries some information on the initial data [25, 26, 27, 28, 29]. *E.g.*, in case of the PNG model with flat initial conditions one finds that (6) is still valid, but the pdf of the random amplitude is given by the Tracy-Widom distribution of the Gaussian orthogonal ensemble (GOE).

The discoveries from 2000 initiated ongoing activities. We refer to a book and reviews [30, 31, 32, 33], where more details can be found. $\diamondsuit$

With the exact solutions obtained since 2000, one has now available a more stringent test for a one-dimensional growth model to be in the KPZ universality class: Not only must be the size of the fluctuations of order $t^{1/3}$, but, for example in the droplet geometry, the random amplitude must be GUE Tracy-Widom distributed. For short, let us call this property the universal one-point pdf. One would expect that any model in the KPZ class shares the universal one-point pdf. In fact, one should regard it as part of the definition for a model to be in the KPZ universality class. So far the evidence comes from the few models which allow for exact computations. One might have guessed some activity from the side of Monte-Carlo simulations. In the early period the statistical sampling was limited. But cumulants

up to the fourth order [16, 17] and even the full height distribution function [34] have been computed for various models, in retrospect achieving surprisingly good agreement with the exact solution in 1+1 dimensions. More recently large scale simulations have been carried out, *e.g.* for off-lattice Eden models in the plane. Exponents are measured with high precision [9], but the full pdf has not yet been determined.

The list of exactly soluble models is short.

● *PNG model, TASEP with discrete and continuous time update, PASEP.* These models are unified in the directed polymer picture. They are zero temperature versions, *i.e.* one studies the fluctuations of the ground state energy of the directed polymer. In the discrete time TASEP the directed polymer lives on the lattice $\mathbb{Z}^2$ with the time direction along $(1, 1)$. The droplet geometry means that both endpoints of the polymer chain are fixed. The random potential $V(i, j)$, $(i, j) \in \mathbb{Z}^2$, is independent, identically distributed with a one-sided geometric distribution, *i.e.* $\text{Prob}(\{V(i, j) = n\}) = (1 - a)a^{|n|}$ for $n \leq 0$ and 0 otherwise with $0 < a < 1$. The PNG model corresponds to the Poisson limit of rare events, $a \ll 1$, while the continuous time TASEP corresponds to the limit of a one-sided exponential.

● *The partially asymmetric simple exclusion process (PASEP).* This is the single step model with a modified rule for updates. As before local minima are increased by 2 at rate p and, in addition, local maxima are deceased by 2 with rate q, $q + p = 1$. The symmetric case $p = q = \frac{1}{2}$ has Gaussian fluctuations. For $p > q$, Tracy and Widom [35] prove the asymptotics of the form (6).

● *KPZ equation.* The KPZ equation (1) has the static exponent $\chi = \frac{1}{2}$, as a consequence of the exactly known steady state, and hence the dynamic exponent $z = 3/2$. To establish the universal one-point pdf requires a detailed computation, as will be explained in the remainder of this contribution. For convenience we set $\lambda > 0$. The initial conditions are the sharp wedge $h(x, 0) = -|x|/\delta$ in the limit $\delta \to 0$. Then, for large t,

$$h(0, t) = -\tfrac{1}{12}(\gamma_t)^3 + 2 \log \alpha + \gamma_t \xi_{\text{TW}} \,, \tag{7}$$

where $\gamma_t = (\alpha^4 \nu t)^{1/3}$. (7) should be compared with (6). The prefactors are the nonuniversal constants specific for the KPZ equation, while the logarithmic shift has its origin in the slightly singular initial condition. In fact, (7) remains valid for every t provided ξ_{TW} is replaced by the random variable ξ_t. Its variance stays bounded as a function of t and its precise pdf is the main result of our contribution.

To close the introduction, we outline some experimental activities.

(i) Ballistic deposition [36]. A rough surface is generated by raining material onto a solid substrate. The randomness results from the incident beam. One difficulty is the suppression of surface diffusion. The incident velocity must be small to ensure a proper local attachment. Also the intensity must be small so to deposit on a relaxed surface. The KPZ fluctuations become visible only in depositing many layers. The surface roughness is measured through small angle scattering of X-rays. With

these severe constraints, at best a qualitative confirmation of the KPZ roughness is obtained.

(ii) Smoldering paper [37]. One fixes a piece of paper in a frame and, after suitable chemical preparation, smolders it from the bottom. It is argued that the one-dimensional combustion front is governed by the KPZ equation. The random noise originates from the intrinsic structure of the paper. Clearly, the geometry corresponds to flat initial conditions and one would like to observe the 1/3 exponent and the GOE Tracy-Widom distribution. To discern such fine statistical properties the experiment has to be repeated many times with identical KPZ parameters, which is difficult to achieve. A total of 18, resp. 21, successful burns are recorded for horizontal, resp. tilted ignition. Using two fitting parameters for the probability density function, reasonable agreement with the theory is obtained.

(iii) Facet edge fluctuations [38]. The experiment is based on the observation that the 1+1 dimensional KPZ equation governs also the thermal fluctuations of the edge of a crystal facet (which is a one-dimensional structure). To measure directly the spatial edge fluctuations is essentially impossible. However one can observe the dynamical fluctuations of the facet edge which via detailed balance are related to the static fluctuation exponent. The experiment clearly rules out random walk like fluctuations of the facet edge, characteristic for an isolated step, and favors the reduced KPZ roughness. Qualitatively the reduction is caused by the constraint through the neighboring step away from the facet and by entropic constraints for deviations into the facet.

(iv) Turbulent liquid crystal [1]. One prepares a thin film of liquid crystal with dimensions 16mm × 16mm × 12μm. The film is confined by two transparent electrodes and subject to oscillating electric fields. A particular point, temperature 25 °C and voltage 26 V at 250 Hz, in the nonequilibrium phase diagram is chosen with care, such that the so-called DSM 2 phase is stable, while the DSM 1 phase is unstable. However the time for spontaneously nucleating the DSM 2 phase in the bulk DSM 1 phase is long compared with the time of a single experimental run. Also unavoidable nucleations at the border of the cell are fairly unfrequent. The system is prepared in the homogeneous DSM 1 phase. A laser pulse plants a seed of the DSM 2 phase which then grows within 35 sec to a droplet filling the region seen by the camera. This is precisely the geometry used in the exact solution of the KPZ equation. To have flat initial conditions the laser pulse plants a line seed. The experiment is repeated of the order of 1000 times. The DSM 1 phase is turbulent, hence good mixing in the region away from the droplet is ensured. For a single probe the physical parameters remain constant over long periods. By spatial isotropy the average shape of the droplet is a disk which improves substantially the analysis of the statistical data. First the nonuniversal constants are determined with good precision. Thereby the scales are fixed and no fitting parameters are used. The scaling exponents, the GUE and GOE Tracy-Widom pdf, and the spatial height-height correlations are measured. Very good agreement with theory is achieved. Also finite

time corrections are determined.

2   The Cole-Hopf solution and its approximations

As written, the KPZ equation is not well-defined mathematically. The noise is singular and even if one takes the smoothening by the Laplacian into account the solutions to the linear part of the equation is rough. Thus the nonlinearity requires to multiply pointwise two distributions which is an ambiguous mathematical operation. The proper definition of solutions to the KPZ equation is ongoing research. For our purposes the Cole-Hopf transform suffices. Of course, it still reflects the difficulties mentioned: the action integral is not well-defined for typical realizations of Brownian motion and of white noise. Fortunately, as in other two-dimensional field theories, an infinite energy renormalization suffices to make sense out of (4). Thus the strategy is to suitably approximate $Z(x,t)$ and then to *define* the solution to the KPZ equation as

$$h(x,t) = (2\nu/\lambda) \log Z(x,t) \,. \tag{8}$$

We will study exclusively the sharp wedge initial conditions, which under Cole-Hopf translates to

$$Z(x,0) = \delta(x) \,. \tag{9}$$

Currently there are five different approximation schemes, which all yield the same limit stochastic process. We discuss each one separately.

(A) Multiple Itô integrals. We expand the exponential of (4) and average over Brownian motion which yields as n-th coefficient

$$\alpha^n \int_{0 \le t_1 \ldots \le t_n \le 2\nu t} dt_1 \ldots dt_n \int_{\mathbb{R}^n} dx_1 \ldots dx_n \prod_{j=1}^n \eta(x_j, t_j) p_0(x_1, t_1, \ldots, x_n, t_n, x, 2\nu t) \tag{10}$$

with p_0 the joint probability density for the Brownian motion to start at 0, to be at x_j at time t_j, $j = 1, \ldots, n$, and to be at x at time $2\nu t$. We now integrate in x and interpret the remaining time integrations as multiple Itô integral in the forward discretization. Then by Itô's lemma

$$\langle Z(x,t)^2 \rangle = \sum_{n=0}^\infty \alpha^{2n} \int_{0 \le t_1 \ldots \le t_n \le 2\nu t} dt_1 \ldots dt_n$$
$$\times \int_{\mathbb{R}^n} dx_1 \ldots dx_n p_0(x_1, t_1, \ldots, x_n, t_n, x, 2\nu t)^2 < \infty \,, \tag{11}$$

since the n-fold integral can be bounded by $C/(n!)^{1/2}$. Hence $Z(x,t)$ is a well defined random variable. In fact, it can be shown that $Z(x,t) > 0$ and that $Z(x,t)$ is continuous in x and in $t > 0$ with probability one [39].

(B) Colored noise. One introduces the mollifier $\varphi_\kappa(x) = \kappa\varphi(\kappa x)$, with $\varphi \ge 0$, φ of

rapid decrease, $\varphi(x) = \varphi(-x)$, $\int dx\varphi(x) = 1$, and spatially smears the white noise to $\eta_\kappa(x,t) = \int dx'\varphi_\kappa(x - x')\eta(x',t)$. The Cole-Hopf transformation remains valid and the action is now a properly defined integral. The white noise average over the partition function is given by

$$\langle Z_\kappa(x,t)\rangle = \langle \mathbb{E}_0\big(\exp\big[\alpha \int_0^{2\nu t} ds\eta_\kappa(b(s),s)\big]\delta(b(2\nu t) - x)\big)\rangle$$
$$= p_0(x, 2\nu t)\exp[\tfrac{1}{2}\alpha^2\varphi_\kappa * \varphi_\kappa(0)t]\,, \tag{12}$$

where $*$ denotes convolution and $p_0(x,t) = (2\pi t)^{-1/2}\exp[-x^2/2t]$ is the transition probability of standard Brownian motion. As $\kappa \to \infty$ one has $\eta_\kappa \to \eta$, but $\langle Z_\kappa(x,t)\rangle$ diverges as $\exp[\tfrac{1}{2}\alpha^2\varphi * \varphi(0)\kappa t]$. On the level of the KPZ equation this means that the average velocity of the interface, $v_\kappa = \tfrac{1}{2}\alpha^2\varphi_\kappa * \varphi_\kappa(0)$, diverges linearly in κ. But it can be proved that in the frame moving with constant velocity v_κ in the h-direction one nevertheless has a well-defined limit as $\kappa \to \infty$. In other words

$$\lim_{\kappa\to\infty} p_0(x,t)Z_\kappa(x,t)/\langle Z_\kappa(x,t)\rangle = Z(x,t) \tag{13}$$

with $Z(x,t)$ as defined in item (A).

(C) Lattice directed polymer. The integral (4) defining the random partition function is discretized. One replaces the Brownian motion $b(t)$ by the random walk ω on the lattice $\mathbb{Z}^2$. ω starts at $(0,0)$ and moves with probability $\tfrac{1}{2}$ either up or right. At each site of the lattice there is independently a unit Gaussian random potential $\eta(i,j)$. Then the discrete approximation to (4) reads

$$Z_N(x,t) = \sum_{\omega:(0,0)\rightsquigarrow((t+x)N,(t-x)N)} 2^{-tN}e^{-\beta E(\omega)}\,, \tag{14}$$

where the sum is only over paths with endpoint $((t + x)N, (t - x)N)$, $|x| < t$, and the energy of the walk ω is defined through

$$E(\omega) = \sum_{(i,j)\in\omega} \eta(i,j)\,. \tag{15}$$

As proved in [40], in the limit $N \to \infty$, $\beta^4 N = \alpha$ fixed,

$$\lim_{N\to\infty} Z_N(x,t) = Z(x,t) \tag{16}$$

as a stochastic process and $Z(x,t)$ as in item (A) with parameters $\nu = \tfrac{1}{2}$ and $D = 1$.

(D) The attractive δ-Bose gas. For integer moments of the partition function with colored noise as in (B) one can carry out the Gaussian average over η with the result

$$\langle Z_\kappa(x,t)^n\rangle = \langle 0|e^{-2\nu t H_n(\kappa)}|x\rangle\,. \tag{17}$$

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On the right one has a matrix element of a quantum propagator for n particles on $\mathbb{R}$. More explicitly

$$H_n(\kappa) = -\sum_{j=1}^{n} \frac{1}{2}\frac{\partial^2}{\partial x_j^2} - \frac{1}{2}\alpha^2 \sum_{i,j=1}^{n} \varphi_\kappa * \varphi_\kappa(x_i - x_j) \tag{18}$$

with $x_j \in \mathbb{R}$ the position of the j-th quantum particle and $|x\rangle$ denoting the quantum state where all n particles are at x. As $\kappa \to \infty$, $Z_\kappa(x,t)/\langle Z_\kappa(x,t)\rangle$ converges to $Z(x,t)/p_0(x,t)$ with $Z(x,t)$ as in (A), while on the right one finds the hamiltonian for n quantum particles with an attractive δ-potential and the self-term omitted. Therefore

$$\langle Z(x,t)^n\rangle = \langle 0|e^{-2\nu t H_n}|x\rangle\,, \tag{19}$$

$$H_n = -\sum_{j=1}^{n} \frac{1}{2}\frac{\partial^2}{\partial x_j^2} - \frac{1}{2}\alpha^2 \sum_{i\neq j=1}^{n} \delta(x_i - x_j)\,. \tag{20}$$

While (19), together with (20), is a correct identity, it cannot be used to define the pdf of $Z(x,t)$, since $\log\langle Z(x,t)^n\rangle \cong n^3$ and the moment problem defined by (19) has many solutions [41].

(E) Weak ASEP. We consider the single step model with asymmetry q, $q > \frac{1}{2}$. The PASEP occupation variables are denoted by $\eta_j(t)$, $j \in \mathbb{Z}$, $\eta_j(t) = 0, 1$. The initial height profile is $h(j,0) = -|j|$. For this initial condition the height at time t is given by

$$h(j,t) = -2 \sum_{\ell=-\infty}^{j} \eta_\ell(t) + j\,. \tag{21}$$

We have seen that for the directed polymer the KPZ equation corresponds to a weak noise approximation. In a similar spirit, the PASEP should approximate the KPZ for weak asymmetry. The asymmetry must be carefully chosen. If it is too weak, one arrives at the Gaussian theory with $\lambda = 0$ and, if the asymmetry is too strong one misses the approximation through a stochastic partial differential equation altogether. The correct choice is as follows: first we scale space-time diffusively, i.e. $j = \lfloor \varepsilon^{-1}x \rfloor$ with $\lfloor \cdot \rfloor$ denoting integer part and time as $\varepsilon^{-2}t$, $t = \mathcal{O}(1)$. $0 < \varepsilon \ll 1$ is the scaling parameter. With this choice we set $q + p = 1$ and $q - p = \beta\sqrt{\varepsilon}$, $\beta > 0$. To recall the ε-dependence of q, we write $h^\varepsilon(j,t)$ instead of $h(j,t)$ in the definition (21). As established in [42, 39], as convergence of stochastic processes it holds

$$\lim_{\varepsilon \to 0} \sqrt{\varepsilon}\beta h^\varepsilon(\lfloor \varepsilon^{-1}x \rfloor, \varepsilon^{-2}t) + \tfrac{1}{2}\beta^2 t\varepsilon^{-1} - \tfrac{1}{24}\beta^4 t - \log(2\sqrt{\varepsilon}/\beta) = \beta h(x,t)\,. \tag{22}$$

Here h on the right side is defined by (8) with $Z(x,t)$ as in item (A) and parameters $\nu = \frac{1}{2}$, $D = \frac{1}{4}$, and $\lambda = \beta$.

The reader might ask, why we make such an extensive list. Firstly, the items provide a better understanding under which physical conditions the KPZ equation

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is a valid approximation. Secondly, the list ensures that the natural but somewhat formal procedure in (A) and (B) properly captures our understanding based on discrete models. Thirdly, and perhaps most importantly, the various approximations do not only give mathematical sense to the KPZ equation but also provide a tool by which, at least in principle, some properties of the KPZ equation can be computed. In this respect, (A) can be used for a short time expansion. (C) has not been of help, yet. But a related discretization, where the directed polymer is placed on $\mathbb{Z} \times \mathbb{R}$, has been used to compute the free energy [43]. The identity in (D) is the gateway to the replica method as used for spin glasses and other disordered systems with the special feature that the matrix element on the right side can be analysed through the Bethe ansatz. While the replicas forces one to work with divergent series, the method has turned out to be a powerful tool. Our exact solution is based on the approach (E). But at an intermediate stage we have to rely on a deep theory for the PASEP developed by Tracy and Widom in recent years [44, 45, 35].

3 Exact solution of the KPZ equation with sharp wedge initial conditions

The pdf of the random free energy $\log Z(x,t)$ with $Z(x,t)$ as in Section 2 has been computed for every x, t. We use a Fredholm determinant formula by Tracy and Widom [35], which in case of 0-1 step initial conditions provides the probability distribution for the position of the m-th particle at time t and thereby the probability distribution of h^ε of (22). While this is a convenient starting point, to actually establish the limit in (22) still requires an intricate asymptotic analysis.

To explain the structure of the exact solution we first recall the pdf for the Tracy-Widom distributed random variable ξ_{TW} of (6). Like the pdf of $h(x,t)$, it is defined in terms of a Fredholm determinant of a symmetric integral operator with kernel over $\mathbb{R} \times \mathbb{R}$. We introduce the Airy kernel

$$K_{\mathrm{Ai}}(x,y) = \int_0^\infty d\lambda \mathrm{Ai}(x+\lambda)\mathrm{Ai}(y+\lambda)\,, \tag{23}$$

where Ai is the standard Airy function. The corresponding operator acting on $L^2(\mathbb{R}, dx)$ is denoted by K_{Ai}. K_{Ai} is a symmetric projection. We also introduce the projection onto the interval $[s, \infty)$, denoted by P_s. $P_s K_{\mathrm{Ai}} P_s$ is trace class for every $s > -\infty$. Then

$$\mathrm{Prob}(\xi_{\mathrm{TW}} \le s) = \det(1 - P_s K_{\mathrm{Ai}} P_s)\,. \tag{24}$$

The Fredholm determinant on the right can be defined through the eigenvalues $\lambda_j(s)$, $j = 1, 2, \ldots$, of $P_s K_{\mathrm{Ai}} P_s$ as

$$\det(1 - P_s K_{\mathrm{Ai}} P_s) = \prod_{j=1}^\infty (1 - \lambda_j(s))\,. \tag{25}$$

In fact, it is more appropriate to think of $P_sK_{\mathrm{Ai}}P_s$ as a large matrix by evaluating as

$$A_{ij}^{(N)} = \chi_s(x_i)K_{\mathrm{Ai}}(x_i, x_j)\chi_s(x_j)\,, \quad i, j = 1, \dots, N\,, \tag{26}$$

with $\{x_j, j = 1, \dots, N\}$ suitably chosen, *e.g.* equally spaced, base points, and χ_s the indicator function of the interval $[s, \infty)$. Then

$$\det(1 - P_sK_{\mathrm{Ai}}P_s) \cong \det(1 - A^{(N)})\,, \tag{27}$$

for large N. The optimal choice of base points and error estimates are discussed in [46].

For the solution of the KPZ equation we obtain

$$(\lambda/2\nu)h(x, t) = -(x^2/4\nu t) - \tfrac{1}{12}(\gamma_t)^3 + 2\log\alpha + \gamma_t\xi_t\,, \tag{28}$$

with $\gamma_t = (\alpha^4\nu t)^{1/3}$, $\alpha = (2\nu)^{-3/2}\lambda D^{1/2}$. We note the self-similar flattening of the deterministic droplet shape proportional to t and a uniform shift, also proportional to t. The relative fluctuations are of order $t^{1/3}$ with a random amplitude ξ_t which is of order 1 but has a pdf changing in time. In (7) we anticipated the special case $x = 0$ and t large. The pdf of ξ_t is given by

$$\rho_t(s) = \int_{-\infty}^{\infty} du\,\gamma_t e^{\gamma_t(s-u)} \exp[-e^{\gamma_t(s-u)}]$$
$$\times \left(\det(1 - P_u(B_t - P_{\mathrm{Ai}})P_u) - \det(1 - P_uB_tP_u) \right)\,. \tag{29}$$

Here B_t has the kernel

$$B_t(x, y) = \int_{-\infty}^{\infty} dw\,(1 - e^{-\gamma_t w})^{-1}\mathrm{Ai}(x + w)\mathrm{Ai}(y + w)\,. \tag{30}$$

and P_{Ai} is an one-dimensional projection with integral kernel $\mathrm{Ai}(x)\mathrm{Ai}(y)$. The first factor under the integral in (29) is the pdf of the Gumbel distribution known from extreme statistics. (29) looks like the convolution of two pdfs. However from numerical simulations we know that the second factor, while normalized, may take negative values.

Equipped with the exact solution (29), we can check whether its long time is given by the GUE Tracy-Widom distribution, to say whether $\xi_t \to \xi_{\mathrm{TW}}$ as $t \to \infty$. Clearly, for $t \to \infty$, the first factor of (29) converges to $\delta(s - u)$ and the second factor to

$$\det(1 - P_u(K_{\mathrm{Ai}} - P_{\mathrm{Ai}})P_u) - \det(1 - P_uK_{\mathrm{Ai}}P_u) = \frac{d}{du}\det(1 - P_uK_{\mathrm{Ai}}P_u)\,, \tag{31}$$

the latter inequality following from identities proved in [47]. We conclude that the universal one-point pdf is established. For very short times ξ_t has a Gaussian distribution of width $t^{1/4}$ [39]. Hence our solution to the KPZ equation describes the crossover from a Gaussian at short times to the GUE Tracy-Widom at long times.

Plots of the pdfs covering mostly late times can be found in [49].

Note. The story behind the exact solution (28), (29) is slightly convoluted. The formula was derived in the fall 2009 by Amir, Corwin, and Quastel [39] and independently by us [48, 49, 50]. All papers use as their starting point a recent Fredholm determinant formula by Tracy and Widom for the PASEP with 0-1 step initial conditions valid for any $q > 1/2$, in particular for $q = \frac{1}{2} + 2\beta\sqrt{\varepsilon}$ [35]. Simultaneously the replica method was persued. Dotsenko and Klumov [51, 52] provide a detailed analysis of the eigenfunctions of the attractive δ-Bose gas on the line. With the hindsight from the exact solution, this leads to

$$\left\langle \exp\left[-\exp(-\lambda + \log Z(x,t) + (x^2/4\nu t) + \tfrac{1}{12}(\gamma_t)^3 - 2\log\alpha)\right]\right\rangle$$
$$= \det(1 - K_{\lambda,t}) \tag{32}$$

for all real λ and $t > 0$, expressing the generating function by a Fredholm determinant [53, 54]. The operator $K_{\lambda,t}$ has the kernel

$$K_{\lambda,t}(x,y) = (1 + e^{(\gamma_t x - \lambda)})^{-1} e^{(\gamma_t x - \lambda)} K_{\mathrm{Ai}}(x,y). \tag{33}$$

Substituting λ by $\gamma_t\lambda$ in (32) and taking the limit $t \to \infty$ on both sides yields indeed the GUE Tracy-Widom distribution function for the rescaled height. Calabrese, Le Doussal, and Rosso [53] establish that if the average in (32) is computed by using (29) one indeed arrives at $\det(1 - K_{\lambda,t})$. $\diamondsuit$

Equipped with such an input the universality of the KPZ equation can be discussed with more precision than before. One aspect concerns how well a microscopic system is approximated by the KPZ equation, which to be meaningful requires a tunable asymmetry. For example, the polynuclear growth model and also the liquid crystal of the experiment would have no such tunable parameter. But if available, then, roughly speaking, the description by the KPZ equation becomes valid in the limit of weak asymmetry and correspondingly long times. An example in case is the 2D Ising model with Glauber dynamics at low temperatures with the $+-$ interface oriented along the (1,1) direction. The asymmetry parameter is the external magnetic field. For weak fields, which is needed anyhow in order to suppress nucleation in the bulk, the motion of the interface is well described by the KPZ equation.

As a second aspect we note that the exact solution of the KPZ equation provides us with qualitative information beyond the universal features. To illustrate we discuss the example of the approach to the Tracy-Widom distribution. In the experiment [1], and also for the TASEP with 0-1 step initial conditions, one finds that the mean is the slowest mode, while higher cumulants decay rapidly to their Tracy-Widom value. Schematically, for large t we write

$$\langle h(0,t)\rangle = v_\infty t + a_1 t^{1/3}(\langle \xi_{\mathrm{TW}}\rangle + c_0 t^{-1/3}) + \mathcal{O}(t^{-1/3}). \tag{34}$$

Here v_∞ and a_1 are model dependent parameters and we discuss the coefficient c_0. $c_0 > 0$ for the experiment and TASEP. Relative to $\langle \xi_{\mathrm{TW}}\rangle$ the mean decays as $t^{-1/3}$

with a positive amplitude. On the other hand one can expand (29) in $1/t$. The expansion is dominated by the Gumbel distribution, which produces a shift by $t^{-1/3}$ to the left. Thus the exponent comes out correctly, but the amplitude and the full first order finite time correction are model dependent. Since the KPZ equation corresponds to weak asymmetry, the PASEP with q close to $1/2$ should have an approach also from the left. This is indeed confirmed by Monte Carlo simulations, which show a sign change of c_0 at $q \approx 0.7$.

4   Conclusions and outlook

By our results we have added one prominent member to the KPZ universality class. Currently more universal pdfs have been computed for lattice models. For curved geometry and for flat initial conditions one knows not only the pdf for a single location but also all multi-point pdfs at the same long time, see [55] for the current status. In addition the case of stationary initial conditions has been studied. For the KPZ equation this would mean to choose an initial height such that $\partial h(x,0)/\partial x$ is white noise in x. One has to see whether the KPZ equation will catch up with these results in the future.

From the point of view of the directed polymer, the continuum KPZ version is the only finite temperature model for which universal pdfs have been obtained so far. It would be of interest to understand how a comparable result can be accomplished for lattice models.

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